

Channel Models & Ch. Capacity

General comm. channel defined in terms of

- 1) possible input X (I/P alphabet)
- 2) possible outputs Y (O/P alphabet)

3) Conditional prob. that ~~states~~ states I/P & O/P sequences of any length n

$$P[Y_1, Y_2, \dots, Y_n | x_1, x_2, \dots, x_n] = P[X \text{ (I/P seq.)}]$$

In a memoryless channel A^n

$$P[Y|X] = \prod_{i=1}^n P[Y_i | x_i]$$

i.e. output at time i depends only on I/P at time i .

S.I.

Binary Symmetric Channel (BSC) Model

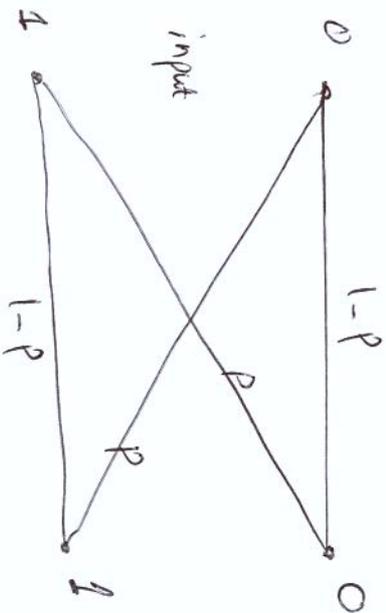
One of simplest channel model
convenient to use

$$X = \{0, 1\}$$

$$Y = \{0, 1\}$$

Appropriate for binary modulation
Hard decisions at detector
Hard decisions at encoder employs binary waveforms

e.g. modulator employs hard decisions
detector makes hard decisions



We can generalize this to +

Discrete Memoryless Channel (DMC)

X is discrete

Y is discrete

Q transition probabilities P_{ij} ~~that determine the probabilities~~
which represents the prob. $P[Y_j | X_i] \triangleq P_{ij}$

The Discrete-Input, Continuous Output Channel:

Suppose that $|X|$ is finite ($|X| = M$)

This channel is

and that $Y = \mathbb{R}$.
again defined by $P(y|x)$ $x \in X, y \in \mathbb{R}$

The most important such channel is

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the AWGN channel for which

$$Y = X + N$$

$$N \sim \mathcal{N}(0, \sigma^2) \Rightarrow P(y|x) \sim \mathcal{N}(x, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

Capacity of the BSC:

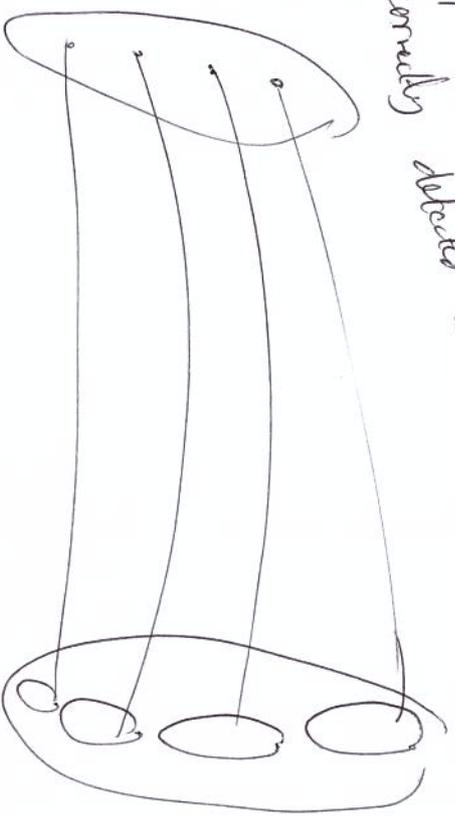
If we send one bit, $P_e = P$

If a seq. of n bits is TX, $P_e = (1-P)^n$

$P_e \rightarrow 0$ as $n \rightarrow \infty$

To improve performance, don't use all binary sequences of length n as possible inputs but to choose a subset & use that subset in it

Subset chosen so that sequences "far apart" from each other as possible so that they can be recognized & corrected even in presence of errors



Assume n bits transmitted

If n is large, then by law of large numbers np bits will be received in error.

So as $n \rightarrow \infty$, probability of receiving np bits in error = 1

Number of seq. of length n that are different from TX seq. at n positions is

$$\binom{n}{np} = \frac{n!}{(np)! (n(1-p))!}$$

Use Stirling's approximation

$$n! \sim \sqrt{2\pi n} n e^{-n}$$

So

$$\binom{n}{np} \approx 2^{n H_b(p)}$$

So if a seq. of length n is transmitted, it is highly probable that one of the $2^{n H_b(p)}$ that are different from the transmitted seq. in np position will be received.

If we insist on using all n sequences, errors are inevitable since there is a considerable overlap between sequences.

However, if we choose a subset of all possible 2^{np} seq.'s & choose subset s.t. the subset of highly probable sequences for each element of this seq. is non-overlapping then reliable comm. is possible.

Total no. of binary seq's = 2^n

$$\Rightarrow \text{we have } M = \frac{2^n}{2^{n H_b(p)}} = 2^{n(1-H_b(p))}$$

seq's of length n transmitted without their corresponding highly probable received sequences overlapping.

So in n uses of channel, we can send M messages.

So, rate or info channel is

$$R = \frac{1}{n} \log_2 M$$

$$R = 1 - H_b(p)$$

R is called the capacity of the channel. R is called the rate for reliable comm. over BSC.

Capacity of Discrete-time AWGN Channel
with E/P power constraint

Channel model:

$$Y_i = X_i + N_i$$

N_i 's are iid Gaussian $\mathcal{N}(0, \sigma^2)$

Input is subject to the power constraint

$$E[X^2] \leq P$$

For large n , we have by the law of large numbers that

$$\frac{1}{n} \|Y\|^2 = E[Y^2] \\ = E[X^2] + E[N^2] \leq P + \sigma^2$$

This means that vector Y is inside n -dim sphere of radius $\sqrt{n(P + \sigma^2)}$

If x is transmitted, the received vector $y = x + n$ satisfies

$$\frac{1}{n} \|y - x\|^2 = \frac{1}{n} \|n\|^2 = \sigma^2$$

~~So~~ y will with high probability y is centered at x with radius $\sqrt{n\sigma^2}$.

The maximum number of spheres of radius $\sqrt{n\sigma^2}$ that

can be packed in a sphere of radius $\sqrt{n(P + \sigma^2)}$

= ratio of the volumes of the two spheres

\Rightarrow maximum number of messages that can be transmitted & still resolved is

$$M = \frac{B_n (\sqrt{n(P + \sigma^2)})^n}{B_n (\sqrt{n\sigma^2})^n} = \left(1 + \frac{P}{\sigma^2}\right)^{n/2}$$

$$B_n =$$

So the maximum rate is

$$R = \frac{1}{h} \log_2 M$$
$$= \frac{1}{2} \log_2 \left(1 + \frac{P}{N_0} \right) \quad \text{bits / transmission}$$

We can also obtain result by maximizing

$I(X; Y)$ over all input pdf's $P(x)$ that satisfy the power constraint $E[X^2] \leq P$

The Capacity of Band-limited Waveform AWGN Channel with an Input Power Constraint

Channel of Bandwidth W is equivalent

to $2W$ uses per second of a discrete AWGN

channel with power constraint $\frac{P}{2W}$ &

noise variance $\sigma^2 = \frac{N_0}{2}$

Capacity of discrete time channel is

$$C = \frac{1}{2} \log_2 \left(1 + \frac{\frac{P}{2W}}{N_0/2} \right)$$
$$= \frac{1}{2} \log_2 \left(1 + \frac{P}{N_0 W} \right) \quad \text{bits/channel use}$$

Capacity of cont-time channel is

$$C = 2W \times \frac{1}{2} \log_2 \left(1 + \frac{P}{N_0 W} \right)$$
$$C = W \log_2 \left(1 + \frac{P}{N_0 W} \right) \quad \text{bits/sec}$$

Calculated eqn. for capacity of a band limited AWGN channel with I/P power constraint derived by Shannon

Remarks:

1) Capacity increases by increasing P
 $C \rightarrow \infty$ as $P \rightarrow \infty$

2) How does BW W affect C?
 Increasing W increases C because we have more transmissions over channel per unit time

Increasing W decreases SNR $\propto \frac{P}{WN_0}$ because increasing W increases effective power entering Receiver

$$\lim_{W \rightarrow \infty} C = \lim_{W \rightarrow \infty} W \log_2 \left(1 + \frac{P}{N_0 W} \right) = \log_2 e \frac{P}{N_0} \text{ bits/sec}$$

So inf BN can not increase capacity indefinitely in contrast to having infinite power

A fundamental relationship between BW & Power efficiency

For reliable comm, we should have

or
 $R < C$
 $R < W \log_2 \left(1 + \frac{P}{N_0 W} \right)$

Non speechlike efficient r is defined by $R \neq R/W$

So
 $r < \log_2 \left(1 + \frac{P}{N_0 W} \right)$

Now, let's write this in terms of E_b/N_0

$$E_b = \frac{E}{\log_2 M} = \frac{PT}{\log_2 M} = \frac{P}{R}$$

So
 $r < \log_2 \left(1 + \frac{P}{N} \frac{R}{W} \right)$
 $\Rightarrow r < \log_2 \left(1 + \frac{E_b}{N} r \right)$

This means that $\frac{E_b}{N_0} > \frac{2^r - 1}{r}$
 Condition for reliable comm in terms of "power efficiency"

Min. E_b/N_0 for which reliable com. is possible is obtained by letting $r \rightarrow 0$ to get $\frac{E_b}{N_0} > \ln 2$. & for this to happen $r \rightarrow 0$ or $W \rightarrow \infty$